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ON SUBCLASSES OF MEROMORPHIC UNIVALENT FUNCTIONS WITH POSITIVE AND FIXED SECOND COEFFICIENTS

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ABSTRACT. In this paper we consider the class $\Sigma^*_{p,c}(\alpha,\beta,\mu)$ consisting of meromorphically univalent functions with positive coefficients and fixed second coefficients. The object of the present paper is to show coefficient estimates and closure theorems for this class. Also we obtain the radius of convexity for functions belonging to the class $\Sigma^*_{p,c}(\alpha,\beta,\mu)$.

KEY WORDS- Univalent, meromorphic, convex.

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1. Introduction

Let Σ denote the class of functions of the form

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty} a_n z^n \tag{1.1}$$

which are regular in $U^* = \{ z : 0 < |z| < 1 \}$ with a simple pole at the origin with residue 1 there. Also let Σ_p denote the class of functions of the form

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty} a_n z^n \qquad (a_n \geq 0) \tag{1.2}$$

that are regular and univalent in U^*.

A function $f(z)$ in Σ_p is in the class $\Sigma_p^*(\alpha,\beta,\mu)$, $0\leq\alpha<1$, $0<\beta\leq1$ and $0\leq\mu\leq1$, if it satisfies for all $z\in U^*$ the condition

$$\left| \frac{\dfrac{zf'(z)}{f(z)}+1}{\mu\dfrac{zf'(z)}{f(z)}-1+(1+\mu)\alpha} \right| < \beta. \tag{1.3}$$

The class $\Sigma_p^*(\alpha,\beta,\mu)$ was studied by Nunokawa, Aouf and Owa [1].

We begin by recalling the following lemma due to Nunokawa, Aouf and Owa [1].

Lemma 1. Let the function $f(z)$ be defined by (1.2). Then $f(z)$ is in the class $\Sigma_p^*(\alpha,\beta,\mu)$ if and only if

$$\sum_{n=1}^{\infty} \{(n+1)+\beta\,[\mu n+(1+\mu)\alpha-1]\}\, a_n \leq (1+\mu)\beta(1-\alpha). \tag{1.4}$$

The result is sharp.

In view of Lemma 1, we can see that the functions f(z) defined by (1.2) in the class $\Sigma_p^*(\alpha,\beta,\mu)$ satisfy the coefficient inequality

$$a_1 \le \frac{(1+\mu)\beta(1-\alpha)}{\{2+\beta[\mu+(1+\mu)\alpha-1]\}} . \qquad (1.5)$$

Hence we may take

$$a_1 = \frac{(1+\mu)\beta(1-\alpha)c}{\{2+\beta[\mu+(1+\mu)\alpha-1]\}} , \qquad 0 \le c \le 1 . \qquad (1.6)$$

Making use of (1.6), we now introduce the following class of functions:

Let $\Sigma_{p,c}^*(\alpha,\beta,\mu)$ denote the subclass of $\Sigma_p^*(\alpha,\beta,\mu)$ consisting of functions of the form

$$f(z) = \frac{1}{z} + \frac{(1+\mu)\beta(1-\alpha)c}{\{2+\beta[\mu+(1+\mu)\alpha-1]\}} z + \sum_{n=2}^{\infty} a_n z^n , \qquad (1.7)$$

where

$a_n \ge 0$, and $0 \le c \le 1$.

In this paper we obtain coefficient inequalities for the class $\Sigma_{p,c}^*(\alpha,\beta,\mu)$ and closure theorems. Further the radius of convexity is obtained for the class $\Sigma_{p,c}^*(\alpha,\beta,\mu)$. Techniques used are similar to those of Silverman and Silvia [2] and Uralegaddi [3].

2. Coefficient Inequalities

THEOREM 1. Let the function f(z) be defined by (1.7). Then f(z) is in the class $\Sigma_{p,c}^*(\alpha,\beta,\mu)$ if and only if

$$\sum_{n=2}^{\infty} \{(n+1)+\beta[\mu n+(1+\mu)\alpha-1]\}a_n \le (1+\mu)\beta(1-\alpha)(1-c). \qquad (2.1)$$

The result is sharp.

Proof. Putting

$$a_1 = \frac{(1+\mu)\beta(1-\alpha)c}{\{2+\beta[\mu+(1+\mu)\alpha-1]\}}, \qquad 0\le c\le 1, \tag{2.2}$$

in (1.4) and simplyifing we get the result. The result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{(1+\mu)\beta(1-\alpha)c}{\{2+\beta[\mu+(1+\mu)\alpha-1]\}} z + \frac{(1+\mu)\beta(1-\alpha)(1-c)}{\{(n+1)+\beta\ [\mu n+(1+\mu)\alpha-1]\}} z^n$$

$$(n \ge 2). \tag{2.3}$$

Corollary 1. Let the function f(z) defined by (1.7) be in the class $\Sigma^*_{p,c}(\alpha,\beta,\mu)$. Then

$$a_n \le \frac{(1+\mu)\beta(1-\alpha)(1-c)}{\{(n+1)+\beta\ [\mu n+(1+\mu)\alpha-1]\}} \qquad (n\ge 2). \tag{2.4}$$

The result is sharp for the function f(z) given by (2.3).

Corollary 2. If $0\le c_1\le c_2\le 1$, then

$$\Sigma^*_{p,c_2}(\alpha,\beta,\mu)\subseteq \Sigma^*_{p,c_1}(\alpha,\beta,\mu).$$

3. Closure Theorems

Theorem 2. Let the functions

$$f_j(z) = \frac{1}{z} + \frac{(1+\mu)\beta(1-\alpha)c}{\{2+\beta[\mu+(1+\mu)\alpha-1]\}} z + \sum_{n=2}^{\infty} a_{n,j}\, z^n \quad (a_{n,j}\ge 0) \tag{3.1}$$

be in the class $\Sigma^*_{p,c}(\alpha,\beta,\mu)$ for every $j=1,2,...,m$. Then the function

$$g(z)=\frac{1}{z}+\frac{(1+\mu)\beta(1-\alpha)c}{\{2+\beta[\mu+(1+\mu)\alpha-1]\}}z+\sum_{n=2}^{\infty}b_n z^n \quad (b_n\geq 0) \tag{3.2}$$

is also in the same class $\Sigma^*_{p,c}(\alpha,\beta,\mu)$, where

$$b_n = \frac{1}{m}\sum_{j=1}^{m} a_{n,j}. \tag{3.3}$$

Proof. Since $f_j(z) \in \Sigma^*_{p,c}(\alpha,\beta,\mu)$ it follows from Theorem 1 that

$$\sum_{n=2}^{\infty}\{(n+1)+\beta[\mu n+(1+\mu)\alpha-1]\}a_{n,j} \leq (1+\mu)\beta(1-\alpha)(1-c) \tag{3.4}$$

for every $j=1,2,...,m$. Hence

$$\sum_{n=2}^{\infty}\{(n+1)+\beta[\mu n+(1+\mu)\alpha-1]\}b_n$$

$$=\sum_{n=2}^{\infty}\{(n+1)+\beta[\mu n+(1+\mu)\alpha-1]\}\left(\frac{1}{m}\sum_{j=1}^{m}a_{n,j}\right)$$

$$=\frac{1}{m}\sum_{j=1}^{m}\sum_{n=2}^{\infty}\{(n+1)+\beta[\mu n+(1+\mu)\alpha-1]\}a_{n,j}$$

$$\leq(1+\mu)\beta(1-\alpha)(1-c) \tag{3.5}$$

and the result follows.

Theorem 3. Let the functions $f_j(z)$ defined by (3.1) be in the class $\Sigma^*_{p,c}(\alpha,\beta,\mu)$ for every $j=1,2,...,m$. Then the function $F(z)$ defined by

$$F(z)=\sum_{j=1}^{m} d_j\, f_j(z) \qquad (d_j \geq 0) \tag{3.6}$$

also in the same class $\Sigma^*_{p,c}(\alpha,\beta,\mu)$, where

$$\sum_{j=1}^{m} d_j = 1 . \tag{3.7}$$

Proof. Combining the definitions (3.1) and (3.6), we have

$$F(z) = \frac{1}{z} + \frac{(1+\mu)\beta(1-\alpha)c}{\{2+\beta[\mu+(1+\mu)\alpha-1]\}} z + \sum_{n=2}^{\infty}\left(\sum_{j=1}^{m} d_j \, a_{n,j}\right) z^n, \tag{3.8}$$

here we have also used the relationship (3.7). Since $f_j(z) \in \Sigma^*_{p,c}(\alpha,\beta,\mu)$ for every $j=1,2,\ldots,m$, Theorem 1 yields

$$\sum_{n=2}^{\infty}\{(n+1)+\beta[\mu n+(1+\mu)\alpha-1]\}a_{n,j} \le (1+\mu)\beta(1-\alpha)(1-c) \tag{3.9}$$

or every $j=1,2,\ldots,m$. Thus we obtain

$$\sum_{n=2}^{\infty}\{(n+1)+\beta[\mu n+(1+\mu)\alpha-1]\}\left(\sum_{j=1}^{m} d_j a_{n,j}\right)$$

$$= \sum_{j=1}^{m} d_j\left(\sum_{n=2}^{\infty}\{(n+1)+\beta[\mu n+(1+\mu)\alpha-1]\}a_{n,j}\right)$$

$$\le (1+\mu)\beta(1-\alpha)(1-c) \tag{3.10}$$

which (in view of Theorem 1) implies that $F(z) \in \Sigma^*_{p,c}(\alpha,\beta,\mu)$.

Theorem 4. The class $\Sigma^*_{p,c}(\alpha,\beta,\mu)$ is closed under convex linear combination.

Proof. Let the functions $f_j(z)$ $(j=1,2)$ defined by (3.1) be in the class $\Sigma^*_{p,c}(\alpha,\beta,\mu)$, it is sufficient to prove that the function $H(z)$

defined by

$$H(z)=\lambda f_1(z)+(1-\lambda)f_2(z) \qquad (0\le\lambda\le1) \tag{3.11}$$

is also in the class $\Sigma^*_{p,c}(\alpha,\beta,\mu)$.

Since

$$H(z)=\frac{1}{z}+\frac{(1+\mu)\beta(1-\alpha)c}{\{2+\beta[\mu+(1+\mu)\alpha-1]\}}z+\sum_{n=2}^{\infty}\{\lambda a_{n,1}+(1-\lambda)a_{n,2}\}z^n, \tag{3.12}$$

we observe that

$$\sum_{n=2}^{\infty}\{(n+1)+\beta[\mu n+(1+\mu)\alpha-1]\}\{\lambda a_{n,1}+(1-\lambda)a_{n,2}\}$$

$$\le (1+\mu)\beta(1-\alpha)(1-c) \tag{3.13}$$

with the aid of Theorem 1. Hence $H(z) \in \Sigma^*_{p,c}(\alpha,\beta,\mu)$. This completes the proof of Theorem 4.

Theorem 5. Let

$$f_1(z)=\frac{1}{z}+\frac{(1+\mu)\beta(1-\alpha)c}{\{2+\beta[\mu+(1+\mu)\alpha-1]\}}z \tag{3.14}$$

and

$$f_n(z)=\frac{1}{z}+\frac{(1+\mu)\beta(1-\alpha)c}{\{2+\beta[\mu+(1+\mu)\alpha-1]\}}z+\frac{(1+\mu)\beta(1-\alpha)(1-c)}{\{(n+1)+\beta[\mu n+(1+\mu)\alpha-1]\}}z^n \qquad (n\ge2). \tag{3.15}$$

Then $f(z)$ is in the class $\Sigma^*_{p,c}(\alpha,\beta,\mu)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty}\lambda_n f_n(z), \tag{3.16}$$

where $\lambda_n \geq 0$ $(n \geq 1)$ and $\sum_{n=1}^{\infty} \lambda_n = 1$.

Proof. We suppose that $f(z)$ can be expressed in the form (3.16). Then we have

$$f(z) = \frac{1}{z} + \frac{(1+\mu)\beta(1-\alpha)c}{\{2+\beta[\mu+(1+\mu)\alpha-1]\}} z + \sum_{n=2}^{\infty} \frac{(1+\mu)\beta(1-\alpha)(1-c)\lambda_n}{\{(n+1)+\beta[\mu n+(1+\mu)\alpha-1]\}} z^n. \tag{3.17}$$

Since

$$\sum_{n=2}^{\infty} \frac{\{(n+1)+\beta\ [\mu n+(1+\mu)\alpha-1]\}}{(1+\mu)\beta(1-\alpha)(1-c)} \cdot \frac{(1+\mu)\beta(1-\alpha)(1-c)\lambda_n}{\{(n+1)+\beta\ [\mu n+(1+\mu)\alpha-1]\}}$$

$$= \sum_{n=2}^{\infty} \lambda_n = 1-\lambda_1 \leq 1, \tag{3.18}$$

it follows from Theorem 1 that $F(z) \in \Sigma^*_{p,c}(\alpha,\beta,\mu)$.

Conversely, we suppose that $f(z)$ defined by (1.7) is in the class $\Sigma^*_{p,c}(\alpha,\beta,\mu)$. Then by using (2.4), we get

$$a_n \leq \frac{(1+\mu)\beta(1-\alpha)(1-c)}{\{(n+1)+\beta\ [\mu n+(1+\mu)\alpha-1]\}} \qquad (n \geq 2). \tag{3.19}$$

Setting

$$\lambda_n = \frac{\{(n+1)+\beta\ [\mu n+(1+\mu)\alpha-1]\}}{(1+\mu)\beta(1-\alpha)(1-c)} a_n \qquad (n \geq 2) \tag{3.20}$$

and

$$\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n, \tag{3.21}$$

we have (3.16). This completes the proof of Theorem 5.

4. Radius of Convexity

Theorem 6. Let the function $f(z)$ defined by (1.7) be in the class $\Sigma^*_{p,c}(\alpha,\beta,\mu)$. Then $f(z)$ is meromorphically convex of order

$\rho (0\le\rho<1)$ in $0<|z|<r=r(\alpha,\beta,\mu,c,\rho)$, where $r(\alpha,\beta,\mu,c,\rho)$ is the largest value for which

$$\frac{(3-\rho)(1+\mu)\beta(1-\alpha)c}{\{2+\beta[\mu+(1+\mu)\alpha-1]\}} r^2 + \frac{n(n+2-\rho)(1+\mu)\beta(1-\alpha)(1-c)}{\{(n+1)+\beta\ [\mu n+(1+\mu)\alpha-1]\}} r^{n+1} \le 1-\rho. \quad (4.1)$$

for $n\ge 2$. The result is sharp for the function

$$f_n(z) = \frac{1}{z} + \frac{(1+\mu)\beta(1-\alpha)c}{\{2+\beta[\mu+(1+\mu)\alpha-1]\}} z + \frac{(1+\mu)\beta(1-\alpha)(1-c)}{\{(n+1)+\beta\ [\mu n+(1+\mu)\alpha-1]\}} z^n \quad \text{for some } n. \quad (4.2)$$

Proof. It is suffices to show that $\left|\frac{zf''(z)}{f'(z)}+2\right|\le 1-\rho$ $(0\le\rho<1)$ for $0<|z|<r(\alpha,\beta,\mu,c,\rho)$. Note that

$$\left|\frac{zf''(z)}{f'(z)}+2\right| \le \frac{\frac{2(1+\mu)\beta(1-\alpha)c}{\{2+\beta[\mu+(1+\mu)\alpha-1]\}} r^2 + \sum_{n=2}^{\infty} n(n+1)a_n\, r^{n+1}}{1 - \frac{(1+\mu)\beta(1-\alpha)c}{\{2+\beta[\mu+(1+\mu)\alpha-1]\}} r^2 - \sum_{n=2}^{\infty} na_n\, r^{n+1}}$$

$$\le 1-\rho \quad (4.3)$$

for $0<|z|<r$ if and only if

$$\frac{(3-\rho)(1+\mu)\beta(1-\alpha)c}{\{2+\beta[\mu+(1+\mu)\alpha-1]\}} r^2 + \sum_{n=2}^{\infty} n(n+2-\rho)a_n\, r^{n+1} \le 1-\rho. \quad (4.4)$$

Since $f(z)$ is in the class $\Sigma^*_{p,c}(\alpha,\beta,\mu)$ from (2.4) we may take

$$a_n = \frac{(1+\mu)\beta(1-\alpha)(1-c)\lambda_n}{\{(n+1)+\beta\ [\mu n+(1+\mu)\alpha-1]\}} \quad (n\ge 2), \quad (4.5)$$

where $\lambda_n\ge 0$ $(n\ge 2)$ and

$$\sum_{n=2}^{\infty} \lambda_n \le 1. \quad (4.6)$$

For each fixed r, we choose the positive integer $n_0=n_0(r)$ for which $\frac{n(n+2-\rho)r^{n+1}}{\{(n+1)+\beta[\mu n+(1+\mu)\alpha-1]\}}$ is maximal. Then it follows that

$$\sum_{n=2}^{\infty} n(n+2-\rho)a_n\, r^{n+1} \leq \frac{n_0(n_0+2-\rho)(1+\mu)\beta(1-\alpha)(1-c)}{\{(n_0+1)+\beta[\mu n_0+(1+\mu)\alpha-1]\}}\, r^{n_0+1}. \tag{4.7}$$

hen $f(z)$ is convex of order ρ in $0<|z|<r(\alpha,\beta,\mu,c,\rho)$ provided that

$$\frac{(3-\rho)(1+\mu)\beta(1-\alpha)c}{\{2+\beta[\mu+(1+\mu)\alpha-1]\}} r^2 + \frac{n_0(n_0+2-\rho)(1+\mu)\beta(1-\alpha)(1-c)}{\{(n_0+1)+\beta[\mu n_0+(1+\mu)\alpha-1]\}} r^{n_0+1} \leq 1-\rho. \tag{4.8}$$

e find the value $r_0=r_0(\alpha,\beta,\mu,c,\rho)$ and the corresponding integer $n_0(r_0)$ o that

$$\frac{(3-\rho)(1+\mu)\beta(1-\alpha)c}{\{2+\beta[\mu+(1+\mu)\alpha-1]\}} r_0^2 + \frac{n_0(n_0+2-\rho)(1+\mu)\beta(1-\alpha)(1-c)}{\{(n_0+1)+\beta[\mu n_0+(1+\mu)\alpha-1]\}} r_0^{n_0+1} = 1-\rho. \tag{4.9}$$

hen this value r_0 is the radius of meromorphically convex of order ρ or functions beloging to the class $\Sigma^*_{p,c}(\alpha,\beta,\mu)$.

References

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